# Infinite Iterated Function Systems

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### Introduction

Let $\Lambda$ be an index set with $|\Lambda|= \aleph_0$, $X \subset \mathbb{R}$ compact and $\Phi = \{\varphi_i : X \to X\}_{i\in\Lambda}$ be a infinite iterated function system (IIFS) of similarities. That is $\varphi_i(x) = r_i(x) + a_i$ for $r_i \in (0,1)$ and $a_i \in \mathbb{R}$. Furthermore assume that $\sup_{i\in\Lambda} r_i < 1$.

We define the encoding space $\Lambda^{\mathbb{N}}$ to be the space of infinite words $\omega = i_1 i_2 i_3 \cdots$. For each $n \in \mathbb{N}$ and $\omega \in \Lambda^{\mathbb{N}}$ we denote the truncated $\omega|_n = i_1 \cdots i_n \in \Lambda^n$ the space of words of length $n$. We denote,

$$\Lambda^* = \bigcup_{n\in\mathbb{N}} \Lambda^n$$

the space of all finite words. To each $\omega \in \Lambda^{\mathbb{N}}$ there corresponds an infinite composition of maps, $\varphi_\omega = \varphi_{i_1} \circ \varphi_{i_2} \cdots$. If $w \in \Lambda^*$ $\varphi_w$ is defined similarly thus giving a meaning to the truncated composition of maps of maps $\varphi_{\omega|_n} = \varphi_{i_1} \circ \cdots \circ \varphi_{i_n}$.

There exists a map $\pi : \Lambda^{\mathbb{N}} \to \mathbb{R}$ defined by,

$$\pi(\omega) = \lim_{n\to\infty} \varphi_{\omega|_n}(x).$$

Hence, we define $K_\Phi$ the limit set of $\Phi$ as

$$K_\Phi = \pi(\Lambda^{\mathbb{N}}).$$

One small note is that this definition seems to depend on $x \in X$. It does not and may actually be replaced by,

$$\pi(\omega) = \lim_{n\to\infty} \varphi_{\omega|_n}(X).$$

If we allow $\varphi_i : \mathbb{R} \to \mathbb{R}$ without the existence of our assumed compact $X$ then there is not a well defined limit set. We note that the limit set satisfies the property that,

$$K_\Phi = \bigcup_{i\in\Lambda} \varphi_i(K_\Phi)$$

In the case that $|\Lambda| < \infty$, there is extensive theory regarding properties of $K_\Phi$ and a simple fact due to Hutchinson [1] is that it is unique and compact. This is not necessarily the case when $|\Lambda|= \aleph_0$. Our first two question involves determining under which conditions $K_\Phi$ is compact and under which is it unique and the dimension of $K_\Phi$ in the case that it is compact and unique.

Additionally we may define a Bernoulli measure $\nu$ on $\Lambda^{\mathbb{N}}$. For each $n \in \mathbb{N}$ and $w \in \Lambda^n$ we define the cylinder as

$$[w] = \{\omega \in \Lambda^{\mathbb{N}} : \omega|_n = w\}.$$

We define $[\varnothing] = \Lambda^{\mathbb{N}}$. Then, let $p = (p_i)_1^\infty$ be a probability vector. We define $\nu = p^{\mathbb{N}}$ as $\nu([w]) = p_w$ for any $w \in \Lambda^*$ where $p_w = p_{i_1} \cdots p_{i_n}$. We may then ask questions about the pushforward of $\nu$ to $K_\Phi$.

Again, in the case that $\Lambda$ is finite this has been very well studied. When $\Lambda$ is countably infinite we may ask whether $\mu = \pi\nu$ is stationary meaning $\mu = \sum_{i\in\Lambda} p_i \cdot \varphi_i \mu$, and questions about the dimension of this measure using various forms of dimension for a measure. Such questions are the remaining four posed.

## 1 Topological Properties of the Attractor

### Question

Find conditions for the uniqueness and compactness of such a limit set $K_\Phi \subset \mathbb{R}$.

### Solution

We begin by showing why the initial condition that $\sup_{i\in\Lambda} r_i < 1$ is necessary. The following example is due to [2].

**Example 1.1.** Let $X = [0,1]$, $\Lambda$ countably infinite, and define $\varphi_i$, $i \in \Lambda$ as follows:

$$\varphi_{i_1}(x) = \frac{x}{2} + \frac{1}{2}, \quad \varphi_{i_n} = \left(1 - 2^{-n}\right) + 2^{-n}x \text{ for } n > 1.$$

We observe that $\sup_{i\in\Lambda}\{r_i\} = 1$. Let $\omega = i_2 i_3 \cdots$. Then,

$$\pi(\omega) = \lim_{n\to\infty} \varphi_{\omega|_n}(X) = \bigcap_{n=1}^{\infty} \varphi_{\omega|_n}([0,1]) = [0, \prod_{n=2}^{\infty}(1 - 2^{-n})].$$

This is not a single point, and thus $\pi : \Lambda^{\mathbb{N}} \to X$ is not well defined. Using the other definition of $\pi$ we note that,

$$\pi(\omega) = \lim_{n\to\infty} \varphi_{\omega|_n}(0) = 0,$$

but also

$$\pi(\omega) = \lim_{n\to\infty} \varphi_{\omega|_n}(1) = \lim_{n\to\infty} \prod_{k=2}^{n}(1 - 2^{-k}) \neq 0.$$

Thus, $\pi$ would depend on the choice of $x \in X$.

We now present an example of why such a compact $X$ must exist in the first place.

**Example 1.2.** Let $\Phi$ be the IIFS of,

$$\varphi_i = \frac{x}{2} + i \quad \text{for } i \in \mathbb{N}.$$

Here, no such $X$ exists and $\varphi_i : [1, \infty) \to [1, \infty)$. For each $n \in \mathbb{N}$ let the letter $i_n$ denote $n$. Let $\omega = i_1 i_2 \cdots$. Then it is clear that the sequence $\varphi_{\omega|_n}(1)$ is unbounded and hence the limit $\pi(\omega)$ does not exist.

Another example due to [2] provides a vexing example of an IIFS without a compact or even unique limit set.

**Example 1.3.** Let $X = [0, 1]$ and $\varphi_i$, $i \in \Lambda$ such that

$$\varphi_{i_1} = \frac{x}{2}, \quad \varphi_{i_n} = 2^{-n}x + 1 - 2^{1-n} \quad \text{for } n > 1.$$

We observe that $K = [0, 1)$. Here, $1 \notin K$ because for any $i \in \Lambda$, $1 \notin \varphi_i[0, 1]$ and hence may not be the image of any $\omega \in \Lambda^{\mathbb{N}}$. Thus, $K$ is not compact. Even worse, $K' = K \setminus \mathbb{Q}$ also suffices as the limit set.

This presents a problem to which there is an easy solution. We simply need to take the closure of $K_\Phi$ and define that to be the attractor. Then, we may find simple conditions for which the identity

$$K_\Phi = \overline{\bigcup_{i \in \Lambda} \varphi_i(K_\Phi)} = \bigcup_{\omega \in \Lambda^{\mathbb{N}}} \bigcap_{n=1}^{\infty} \varphi_{\omega|_n}(K_\Phi)$$

holds.

We will need a quick lemma stated without proof.

**Lemma 1.4.** If $X$ is a complete and compact metric space then $\mathcal{K}(X)$ the space of all non-empty compact subsets of $X$ endowed with the Hausdorff metric is complete and compact. $\square$

Now, the conditions for $K_\Phi$ to be compact are as follows.

**Theorem 1.5.** Let $\Phi = \{\varphi_i : \mathbb{R} \to \mathbb{R}\}_{i \in \Lambda}$ be an IIFS with $|\Lambda| = \aleph_0$. Furthermore, suppose each $\varphi_i$ is a similarity meaning $\varphi_i(x) = r_i x + a_i$ for $r_i \in (0, 1)$ and $a_i \in \mathbb{R}$. Suppose there exists $\varepsilon > 0$ such that $\sup_{i \in \Lambda} r_i \leq 1 - \varepsilon$ and $\sup_{i \in \Lambda} |a_i| < \varepsilon^{-1}$. Then $K_\Phi$ as defined is compact.

*Proof.* This is a relatively straightforward application of Banach's Fix Point Theorem.

Let $a = \sup_{i \in \Lambda} |a_i|$ and $R = \sup_{i \in \Lambda} r_i$. We know that $a < \varepsilon^{-1}$ so it is finite, and $R < 1 - \varepsilon$. Thus the set

$$X = \left[\frac{-a}{1-R}, \frac{a}{1-R}\right]$$

is compact (and complete). Furthermore $\varphi_i(X) \subseteq X$ for all $i \in \Lambda$. This is because, for each $x \in X$ and $i \in \Lambda$

$$|\varphi_i(x)| = |r_i x + a_i| \leq |Ra + a_i| \leq (R+1)a.$$

By the Lemma above $\mathcal{K}(X)$ is complete and compact. Now consider the set mapping $F : \mathcal{K}(X) \to \mathcal{K}(X)$ defined by

$$F(K) = \overline{\bigcup_{i \in \Lambda} \varphi_i(K)}.$$

Equipping $\mathcal{K}(X)$ with the Hausdorff metric, $d_H$, we see that $F$ is a contraction because

$$d_H\left(F(K_1), F(K_2)\right) \leq R d_H\left(K_1, K_2\right).$$

Hence, Banach's Fixed Point Theorem ensures that $F$ has a fixed point which is precisely $K_\Phi$. $\square$

## 2 Dimension of the Attractor

### Questions

Assume $K_\Phi$ exists and let

$$\dim_s \Phi = \inf \left\{ t \in \mathbb{R} : \sum_{i \in \Lambda} r_i^t \leq 1 \right\}.$$

Is it always true that $\dim_H K_\Phi \leq \dim_s \Phi$? Under what conditions on $\Phi$ does $\dim_H K_\Phi = \min\{1, \dim_s \Phi\}$. (Here $\dim_H$ denotes the Hausdorff dimension of a subset of $\mathbb{R}$.)

### Solution

Denote the $s$ dimensional Hausdorff measure as $\mathcal{H}^s$. We first show that $K_\Phi$ is $\mathcal{H}^s$ measurable for each $s \geq 0$. To do this note that

$$K_\Phi = \bigcup_{\omega \in \Lambda^{\mathbb{N}}} \bigcap_{n=1}^{\infty} \varphi_{\omega|_n}(K_\Phi).$$

We appeal to some descriptive set theory and note that $K_\Phi$ is a Suslin set, and thus are $\mathcal{H}^s$-measurable [3].

With the measurability of these sets ensured, we now investigate quantitative information on their dimension. First, we state without proof a basic definition and lemma.

**Definition 2.1.** Let $|U|= \operatorname{diam}(U)$. For any $E \subset \mathbb{R}$ we write measurable we write

$$\mathcal{H}^s_\infty(E) = \inf \left\{ \sum_{n=1}^{\infty} |U_n|^s : \ \{U_n\}_1^\infty \text{ is a cover for } E \right\}.$$

**Lemma 2.2.** Given $E \subset \mathbb{R}$, $\dim_H(E) = \inf\{s \geq 0 : \mathcal{H}^s_\infty(E) = 0\}$ □

We will also need the following analytic lemma

**Lemma 2.3.** Let $n \in \mathbb{N}$ be a finite integer and suppose $\{a_{i,k_i}\}_{k_i \in \mathbb{N}, 1 \leq i \leq n}$ is a collection of $n$ sequences whose sum converges absolutely. Then, the $n$-fold Cauchy product converges:

$$\prod_{i=1}^{n} \left( \sum_{k_i=0}^{\infty} a_{i,k_i} \right) = \lim_{N \to \infty} \sum_{k_1 + \cdots + k_i \leq N} a_{1,k_1} \cdots a_{i,k_i}.$$

□

We now prove the affirmative of the first question in this section.

**Theorem 2.4.** Let $K$ be the attractor of an IIFS of similarities $\Phi$ such that there exists $\varepsilon > 0$ with $0 < r_i < 1 - \varepsilon$ and $|a_i| < \varepsilon^{-1}$ for all $i \in \Lambda$. Furthermore suppose there exists $s_0 > 0$ such that $\sum_{i \in \Lambda} r_i^{s_0} < \infty$. Then,

$$\dim_s \Phi \geq \dim_H(K).$$

*Proof.* Using the result of Lemma 2.2 it suffices to show that for any $t > \alpha = \dim_s \Phi$, $\mathcal{H}^t_\infty(K) = 0$. By the definition of $\alpha$, $\sum_{i \in \Lambda} r_i^t < 1$. By the defining property of the attractor, for any $n \in \mathbb{N}$,

$$K \subseteq \bigcup_{w \in \Lambda^n} \varphi_w(K).$$

Thus per the definition of $\mathcal{H}^s_\infty$

$$\mathcal{H}^t_\infty(K) \leq \sum_{w \in \Lambda^n} |\varphi_w(K)|^t = |K|^t \sum_{w \in \Lambda^n} r_w^t.$$

We may use Lemma 2.3 to bound the right hand side of the above.The contraction ratios corresponding to words $w = i_1, \cdots i_n$ are in bijection with the terms in the n-fold Cauchy product,

$$|K|^t \sum_{w \in \Lambda^n} r_w^t = |K|^t \left( \sum_{N=1}^{\infty} \left( \sum_{i_1 + \cdots + i_k \leq N} r^t_{1,i_1} \cdots r^t_{k,i_k} \right) \right) = |K|^t \left( \sum_{i \in \Lambda} r_i^t \right)^n$$

Because $\sum_{i \in \Lambda} r_i^t < 1$, taking $n \to \infty$ we see that $\mathcal{H}_\infty(K)^t = 0$. □

With this resolved we can prove a stronger statement answering the second question posed in this section. First, a couple definitions for the conditions we will need.

**Definition 2.5.** An IIFS (or IFS) $\Phi$ has exact overlaps if there exist distinct $w_1, w_2 \in \Lambda^*$ with $\varphi_{w_1} = \varphi_{w_2}$.

**Definition 2.6.** An IIFS (or IFS) $\Phi$ is said to be algebraic if $\{r_i, a_i\}_{i\in\Lambda}$ are algebraic numbers.

Now, we state a result due to Hochman [4] which we will shortly use.

**Theorem 2.7.** For a finite algebraic IFS $\Phi$ of similarities of $\mathbb{R}$ without exact overlaps,

$$\dim_H(K) = \min\{1, \dim_s \Phi\}$$

This is actually due to a stronger result about IFS satisfying what is called the exponential scalar condition, but we will not use that here. We now state and prove the main theorem of this section about the dimension of attractors of IIFS.

**Theorem 2.8.** Suppose $\Phi$ is an algebraic IIFS of similarities of $\mathbb{R}$ such that there exists $\varepsilon > 0$ with $0 < r_i < 1 - \varepsilon$ and $|a_i| < \varepsilon^{-1}$ for all $i \in \Lambda$. Furthermore suppose it has no exact overlaps and there exists $s_0 > 0$ such that $\sum_{i\in\Lambda} r_i^{s_0} < \infty$. Then,

$$\dim_H(K) = \min\{1, \dim_s \Phi\}.$$

*Proof.* Let $\alpha = \min\{1, \dim_s \Phi\}$. It suffices to show that for all $\delta > 0$ there exists a finite $\Lambda' \subset \Lambda$ such that

$$\dim_s \Phi' \geq \alpha - \delta.$$

Using the exact dimensionality result of Theorem 1.7, the above inequality is equivalent to

$$\dim_H K'_\Phi = \min\{1, \dim_s \Phi'\} \geq \min\{1 - \delta, \dim_s \Phi - \delta\} = \alpha - \delta.$$

The statement then follows from $\dim_H(K_\Phi) \geq \dim_H(K_{\Phi'})$ as $K_\Phi \supseteq K_{\Phi'}$.

To this end let $\delta > 0$ be arbitrarily small and let $s = \dim_s \Phi$. We proceed by cases depending on the value of $s$, proving the theorem in each case.

If $s > 1$ then, $\sum_{i\in\Lambda} r_i > 1$. Indeed, for any $\Lambda' \subset \Lambda$, $\sum_{i\in\Lambda'} r_i^s < \sum_{i\in\Lambda} r_i$. Thus, there exists some $\Lambda' \subset \Lambda$ that is finite, such that $\sum_{i\in\Lambda'} r_i > 1$ implying that $\dim_s \Phi' > 1$ where $\Phi'$ is the IFS associated with $\Lambda'$. By Theorem 1.7, $\dim_H K_{\Phi'} = 1$ implying that $\dim_s K_\Phi = 1$ as well. Hence, $\dim_H K_\Phi = \min\{1, s\}$.

Now, suppose that $s \leq 1$. Let $\delta > 0$, fix $i_0 \in \Lambda$ and choose $\Lambda' \subset \Lambda$ finite with $i_0 \in \Lambda'$. We define $f : \mathbb{R} \to \mathbb{R}$ as

$$f(t) = \sum_{i\in\Lambda'} r_i^t.$$

Observe that $f$ is a smooth convex function. Using our assumptions,

$$1-\delta \leq f(s) = \sum_{i\in\Lambda'} r_i^s < 1.$$

Additionally, as before $s' < 1$ and also $s' < s$. We observe that $f$ is a smooth function and has a smooth inverse. Using basic calculus,

$$(f^{-1})'(t) = \frac{1}{f'(f^{-1}(t))}.$$

Because $f^{-1}$ is smooth it is Lipschitz and a sufficient Lipschitz constant given the domain of $[1-\delta, \delta]$.

$$\sup_{1-\delta\leq t\leq\delta} |(f^{-1})'(t)| = \sup_{1-\delta\leq t\leq\delta} \left|\frac{1}{f'(f^{-1}(t))}\right| = \left|\frac{1}{f'(f^{-1}(1))}\right| = \left|\frac{1}{f'(s')}\right|.$$

The last two equalities follows from the fact that since $f$ is strictly decreasing, $f^{-1}$ is too, and $f'$ is strictly increasing. Thus, the supremum occurs when $f'(f^{-1}(t))$ is minimized, and since $f^{'}$ is strictly increasing this is when $f^{-1}(t)$ is minimized which occurs when $t$ is maximized. Because each term $\log(r_i)r_i$ is negative,

$$|f'(s')| = \left|\sum_{i\in\Lambda'} \log(r_i) r_i^{s'}\right| \geq \left|\log(r_{i_0}) r_{i_0}^{s'}\right| \geq |\log(r_{i_0}) r_{i_0}| = M.$$

This implies that

$$\left|\frac{1}{f'(s')}\right| \leq M^{-1}.$$

Thus, using this as a Lipschitz constant,

$$|f^{-1}(x) - f^{-1}(y)| \leq M^{-1}|x-y|$$

for all $x, y \in [1-\delta, 1]$. Taking $x = 1$ and $y = \sum_{i\in\Lambda'} r_i^s$,

$$\left|f^{-1}(x) - f^{-1}(y)\right| = |s'-s| \leq M^{-1}\left|1 - \sum_{i\in\Lambda'} r_i^s\right| \leq M^{-1}\delta.$$

Because $s' < s$ we have $s' > s - M^{-1}\delta$. Using Theorem 1.7 and the fact that $s' < 1$ we have,

$$\dim_H K_\Phi \geq \dim_H K_{\Phi'} = \dim_s \Phi' \geq \dim_s \Phi - M\delta.$$

Because $M^{-1}$ is independent of $\delta$ we may conclude that $\dim_H K_\Phi = \dim_s \Phi$ as we take $\delta \to 0$.

We have thus shown in each case of $s$ that $\dim_H K_\Phi = \min\{1, \dim_s \Phi\}$. □

What follows is a much simpler proof.

*Proof.* Fix $0 < \varepsilon < 1$ and $i_0 \in \Lambda$. Let $s = \dim_s \Phi$. We again show that there exists finite $\Lambda' \subset \Lambda$ and associated IFS $\Phi'$ such that $\dim_s \Phi' > s - \varepsilon$.

This suffices as,

$$\dim_H K_\Phi \geq \dim_H K_{\Phi'} = \min\{1, s'\} > \min\{1, s\} - \varepsilon.$$

Taking $\varepsilon \to 0$ and thus $\Lambda' \to \Lambda$ shows that $\dim_H K_\Phi = \min\{1, s\}$ as desired.

Returning to the problem at hand we let $0 < \eta, \delta < 1$ be parameters with $\eta \ll \varepsilon$ and such that $r_{i_0}^s (r_{i_0}^{-\eta} - 1) > 2\delta$.

Take finite $\Lambda' \subset \Lambda$ such that $i_0 \in \Lambda'$ and $\sum_{i \in \Lambda'} r_i^s > 1 - \delta$. Then,

$$\begin{aligned}\sum_{i \in \Lambda'} r_i^{s-\eta} &> \sum_{i \in \Lambda'} r_i^s + \left(r_{i_0}^{s-\eta} - r_{i_0}^s\right) \\ &> 1 - \delta + r_{i_0}^s (r_{i_0}^{-\eta} - 1) \\ &> 1 + \delta\end{aligned}$$

Thus, there exists $s' \in [s - \eta, s]$ such that $\sum_{i \in \Lambda'} r_i^{s'} = 1$ which is precisely $s' = \dim_s \Phi'$. Because $\eta \ll \varepsilon$, and $s > s'$ necessarily, $s - s' < \varepsilon$. This is equivalent to,

$$s' = \dim_s \Phi' > s - \varepsilon.$$

$\square$

## 3 Existence and Uniqueness of Stationary Measures

### Question

Let $(p_i)_{i \in \Lambda}$ be a probability vector. Find conditions for the existence and uniqueness of $(\Phi, p)$-stationary measures.

### Solution

Let $(p_i)_{i \in \Lambda}$ be a probability vector and let $\Lambda^{\mathbb{N}}$ be the encoding space. Suppose $\Phi$ the IIFS satisfies the conditions for the uniqueness and compactness of a limit set $K_\Phi$ as stated in Section 1.

Set $X \subseteq \mathbb{R}$ be the compact subset such that $\varphi_i(X) \subset X$ for all $i \in \Lambda$. Let $r = \sup_{i \in \Lambda}\{r_i\} < 1$. For each $\omega \in \Lambda^{\mathbb{N}}$, $\varphi_{\omega|_n}(X) \subseteq \varphi_{\omega|_{n-1}}(X)$ and

$$\operatorname{diam}\left(\varphi_{\omega|_n}(X)\right) \leq r^n \operatorname{diam}(X).$$

This approaches 0 as $n \to \infty$. Taking $\pi : \Lambda^{\mathbb{N}} \to \mathbb{R}$ as in the introduction, we see it is well defined. Give $\Lambda^{\mathbb{N}}$ the metric

$$\rho(\omega, \eta) = \begin{cases} 2^{\min\{n \geq 0 \,:\, \omega|_n \neq \eta|_n\}} & \text{if } \omega \neq \eta \\ 0 & \text{if } \omega = \eta \end{cases}$$

**Theorem 3.1.** $\pi$ is Hölder continuous with respect to the standard Euclidean metric $|\cdot|$ on $\mathbb{R}$ and $\rho$ on $\Lambda^{\mathbb{N}}$.

*Proof.* To prove this we introduce $\sigma : \Lambda^{\mathbb{N}} \to \Lambda^{\mathbb{N}}$ the left shift map so $\sigma(\omega)|_n = \omega|_{n+1}$. Let $\omega, \eta \in \Lambda^{\mathbb{N}}$ and $m = \min\{n \geq 0 : \omega|_n \neq \eta|_n\}$

$$\begin{aligned}|\pi(\omega) - \pi(\eta)| &= |\varphi_{\omega|_m}\pi(\sigma^m(\omega)) - \varphi_{\eta|_m}\pi(\sigma^m(\eta))| \\ &\leq \operatorname{diam}(X)r^m = \operatorname{diam}(X)2^{m\log_2(r)} = \operatorname{diam}(X)\rho(\omega,\eta)^{\log_2 r}.\end{aligned}$$

□

We now define $\nu$ to the Bernoulli measure on $\Lambda^{\mathbb{N}}$ induced by $(p_i)_{i\in\Lambda}$ as in the introduction.

**Theorem 3.2.** The measure $\mu = \pi\nu$ is $(\Phi, p)$-stationary that is,

$$\mu = \sum_{i\in\Lambda} p_i \cdot \varphi_i\mu.$$

*Proof.* For $i \in \Lambda$ let $\psi_i : \Lambda^{\mathbb{N}} \to \Lambda^N N$ be the map such that $\psi(\omega) = i\omega$. For $n \geq 1$, $i_1 \cdots i_n = w \in \Lambda^n$,

$$\begin{aligned}\sum_{i\in\Lambda} p_i \cdot \psi_i\nu[\omega] &= \sum_{i\in\Lambda} p_i\nu\{\omega : i\omega \in [w]\} \\ &= p_{i_1} \cdot \nu\{\omega : i_1\omega \in [w]\} \\ &= p_{i_1} \cdot \nu[i_2 \cdots i_n] = p_{i_1} \cdots p_{i_n} = p_w.\end{aligned}$$

Because this holds for arbitrary $w \in \Lambda^n$ and $i \in \Lambda$ by the uniqueness of $\nu$ we have that $\nu = \sum_{i\in\Lambda} p_i \cdot \psi_i\nu$. Thus,

$$\mu = \pi\nu = \sum_{i\in\Lambda} p_i \cdot \pi\psi_i\nu = \sum_{i\in\Lambda} p_i \cdot \varphi_i\pi\nu = \sum_{i\in\Lambda} p_i \cdot \varphi_i\mu.$$

□

**Theorem 3.3.** If $\mu$ is a $(\Phi, p)$-stationary measure than it is supported in $K_\Phi$.

*Proof.* We begin by noting that $\mu$ is a Borel probability measure on $\mathbb{R}$ and hence is outer regular. It suffices to show that for any open $K_\Phi \subset V \subset X$ that $\mu(V) = 1$.

For $n \geq 1$ we define

$$E_n = \{x \in X : \varphi_w(x) \in V \text{ for all } w \in \Lambda^n\}.$$

Let $F : \mathcal{K}(X) \to \mathcal{K}(X)$ be defined as in the proof of Theorem 1.5. Then, for each $x \in X$, $F^n(\{x\}) = \{\varphi_w(\{x\})\}_{w\in\Lambda^n}$ and this converges to $K_\Phi$. Hence, there exists $N \in \mathbb{N}$, dependent on $x$, such that for any $n > N$, $F^n(\{x\}) \in E_n$. We observe that

$$X = \bigcup_{N\geq 1} \bigcap_{n\geq N} E_n.$$

Thus, $\lim_{n\to\infty}\mu(E_n)=1$. For each $n\geq 1$,

$$\mu(V)=\sum_{w\in\Lambda^n}p_w\cdot\pi\mu(\varphi_w^{-1}(V))\geq\sum_{w\in\Lambda^n}p_w\mu(E_n)=\mu(E_n).$$

Taking $n\to\infty$, $\mu(V)=1$ proving the theorem. □

**Theorem 3.4.** There exists a unique $(\Phi,p)$-stationary measure (on $K_\Phi$).

*Proof.* Suppose that $\mu_1$ and $\mu_2$ are both stationary measures. By the previous result, they are both supported on $K_\Phi$. Let $g:K_\Phi\to\mathbb{R}$ be continuous and let $\varepsilon>0$. Because $K_\Phi$ is compact, $g$ is uniformly continuous and so there exists $\delta>0$ such that $|g(x)-g(y)|<\varepsilon$ for all $x,y\in K_\Phi$, $|x-y|<\delta$.

Let $n\geq 1$ be such that $\operatorname{diam}(\varphi_w(K_\Phi))<\delta$ for all $w\in\Lambda^n$. Given $x_0\in K_\Phi$,

$$\begin{aligned}\left|\int gd\mu_1-gd\mu_2\right| &\leq\sum_{w\in\Lambda^n}p_w\cdot\left|\int gd(\varphi_w\mu_1)-g\circ\varphi_w(x_0)+g\circ\varphi_w(x_0)-\int gd(\varphi_w\mu_2)\right|\\ &\leq\sum_{k=1,2}\sum_{w\in\Lambda^n}p_w\cdot|g\circ\varphi_w(x)-g\circ\varphi_w(x_0)d\mu_k(x)|\\ &\leq\sum_{k=1,2}\sum_{w\in\Lambda^n}p_w\cdot\varepsilon=2\varepsilon.\end{aligned}$$

Taking $\varepsilon\to 0$ we see that $\int g\mu_1=\int g\mu_2$ and so $\mu_1\equiv\mu_2$. □

Theorems 3.2 and 3.4 show that our candidate $\mu=\pi\varphi$ is the unique $(\Phi,p)$-stationary measure on $K_\Phi$. These results were adapted from [5]

# 4 An Upper Bound and Exactness of the Dimension of Self Similar Measures

## 4 Background Definitions and Results

We first recall some definitions and results which will be important throughout this section.

**Definition 4.1.** If $p=(p_i)_{i\in\Lambda}$, we denote the entropy of a probability vector as

$$H(p)=-\sum_{i\in\Lambda}p_i\log(p_i).$$

We also have the Lyupanov Exponent.

**Definition 4.2.** The Lyupanov Exponent, $\chi(\Phi,p)$ is defined as

$$\chi(\Phi,p)=-\sum_{i\in\Lambda}p_i\log(r_i).$$

We now define some notions of dimensionality of a Borel measure $\mu \in \mathcal{M}(\mathbb{R})$.

**Definition 4.3.** For $\mu \in \mathcal{M}(\mathbb{R})$ and $x \in \mathbb{R}$ we define the lower local dimension of $\mu$ as

$$\underline{\dim}(\mu, x) = \liminf_{r \searrow 0} \frac{\log \mu(B(x,r))}{\log(r)}.$$

The following standard result is known as the Mass Distribution Principle and we state, without proof, a specific case for Borel measures on $\mathbb{R}$.

**Theorem 4.4** (Mass Distribution Principle)**.** Let $\mu \in \mathcal{M}(\mathbb{R})$, $E \in \mathcal{B}(\mathbb{R})$ with $\mu(E) > 0$ and $s \geq 0$. Then, $\dim_H(E) \geq s$ if $\underline{\dim}(\mu, x)$ for all $x \in E$. □

The next result is due to Billingsley and carries his namesake.

**Lemma 4.5** (Billingsley's Lemma)**.** Let $\mu \in \mathcal{M}(\mathbb{R})$, $E \in \mathcal{B}(\mathbb{R})$ with $\mu(E) > 0$ and $s \geq 0$. Then $\dim_H(E) \leq s$ if $\underline{\dim}(\mu, x) \leq s$ for all $x \in E$. □

We need the following Lemma before we define the Hausdorff dimension of a measure.

**Lemma 4.6.** Given $\mu \in \mathcal{M}(\mathbb{R})$ and $r > 0$. The function $x \mapsto \mu(B(x,r))$ is Borel measurable and

$$\underline{\dim}(\mu, x) = \liminf_{n \to \infty} \left\{ \frac{\log \mu(B(x,r))}{\log r} : r \in \mathbb{Q} \cap (0,1). \right\}$$

Hence, we may define the Hausdorff dimension of a measure $\mu$ as follows.

**Definition 4.7.** For $\mu \in \mathcal{M}(\mathbb{R})$ we define

$$\dim_H \mu = \sup \left\{ s \geq 0 \ : \ \underline{\dim}(\mu, x) \geq s \text{ for a.e. } x \in \mathbb{R} \right\}.$$

It has an equivalent and perhaps easier to remember definition.

**Theorem 4.8.**

$$\dim_H \mu = \inf \left\{ \dim_H E \ : \ E \in \mathcal{B}(\mathbb{R}), \mu(E) > 0 \right\}.$$

*Proof.* Let the right hand side by $d$. We show both $\dim_H \mu \leq d$ and $\dim_H \mu \geq d$.

Let $E \in \mathcal{B}$ such that $\mu(E) > 0$. By the definition of $\dim_H \mu$ we observe that $\underline{\dim}(\mu, x) \geq \dim_H \mu$ for a.e. $x \in \mathbb{R}$. Hence, there exists $E_0 \subset E$ also Borel such that $\mu(E_0) = \mu(E) > 0$ and $\underline{\dim}(\mu, x) \geq \dim_H(\mu)$ for all $x \in E_0$. It follows by the Mass Distribution Principle, $\dim_H E \geq \dim_H(\mu)$. Thus $\dim_H \mu \leq d$.

Suppose $s > \dim_H \mu$. We take $E = \{x \in \mathbb{R} : \underline{\dim}(\mu, x) \leq s\}$. By the previous lemma, $E$ is Borel and $\mu(E) > 0$ per the definition of $\dim_H \mu$. By Billinglsey's Lemma, $\underline{\dim}(\mu, x) \leq s$ for $x \in E$ implies that $\dim_H E \leq s$ and so $d \leq s$ or equivalently, $\dim_H \mu \geq d$. □

We shall also need the following definition which is the last of this introductory section.

**Definition 4.9.** We say $\mu \in \mathcal{M}(\mathbb{R})$ is exact dimensional if there exists a nonnegative real number say $\dim \mu$ which we call the dimension of $\mu$, so that the following limit exists and,

$$\lim_{r \searrow 0} \frac{\log \mu(B(x,r))}{\log(r)} = \underline{\dim}(\mu, x) = \dim \mu$$

for a.e. $x \in \mathbb{R}$.

**Question**

Assume that $\Phi$ is an IIFS of similarities and $p = (p_i)_{i \in \Lambda}$ a probability vector. Let $\mu = \pi\nu$ the pushforward of $\nu$ the probability on $\Lambda^{\mathbb{N}}$ induced by $p$ by the encoding map $\pi$. Thus $\mu$ is $(\Phi, p)$ stationary as proved in Section 3. Furthermore, assume that $H(p) < \infty$ and $\chi(\Phi, p) < \infty$. Is it true that

$$\dim_H(\mu) \leq \frac{H(p)}{\chi(\Phi, p)} = \dim_s(\Phi, p)?$$

Additionally, is it true that $\mu$ is exact dimensional?

**Solution**

A note before we begin is that the above two questions have been answered in the affirmative in the case that $\Phi$ is a finite IFS. We simply adapt these proofs to the IIFS case. We introduce some notation. For $\omega \in \Lambda^{\mathbb{N}}$

$$\alpha(\omega) = \limsup_{\delta \searrow 0} \frac{\log \mu(B(\pi\omega, \delta))}{\log \delta}.$$

**Lemma 4.10.** Let $\omega \in \Lambda^{\mathbb{N}}$ and let $\{u_n\}_1^\infty \subset (0, \infty)$ be a strictly decreasing sequence with $\limsup_{n\to\infty} \frac{u_{n+1}}{u_n} < \infty$. Then,

$$\alpha(\omega) = \limsup_{n\to\infty} \frac{\log \mu(B(\pi\omega, u_n))}{\log u_n}.$$

$\square$

**Theorem 4.11.** For $\mu$ a.e. $x \in \mathbb{R}$,

$$\limsup_{\delta \searrow 0} \frac{\log \mu(B(x, \delta))}{\log \delta} \leq \dim_s(\Phi, p) = s.$$

*Proof.* Denote $\nu = p^{\mathbb{N}}$ the Bernoulli measure as described in the introduction. Because $\mu = \pi\nu$ where $\pi$ is the encoding map, it suffices to show for all $\omega \in \Lambda^{\mathbb{N}}$, and $\{u_n\}$ to be defined later,

$$\alpha(\omega) = \limsup_{n\to\infty} \frac{\log \mu(B(\pi\omega, u_n))}{\log u_n} \le s.$$

We let $R = \operatorname{diam}(K)$ and $\sigma : \Lambda^{\mathbb{N}} \to \Lambda^{\mathbb{N}}$ the left shit map. Because $\varphi_{\omega|_n}(\pi\sigma^n\omega) = \pi\omega$, $\pi\omega \in \varphi_{\omega|_n}(K)$. Thus,

$$\pi\omega \in B(\pi\omega, Rr_{\omega|_n}).$$

Thus, we define $u_n = Rr_{\omega|_n}$ and note that it satisfies the conditions of the lemma. Additionally,

$$\alpha(\omega) = \limsup_{n\to\infty} \frac{\log \mu(B(\pi\omega, Rr_{\omega|_n}))}{\log Rr_{\omega|_n}} \le \limsup_{n\to\infty} \frac{\log \mu(\varphi_{\omega|_n}(K))}{\log r_{\omega|_n}}.$$

We now consider the cylinders $[\omega|_n]$. For every $\eta \in [\omega|_n]$, $\pi\eta \in \varphi_{\omega|_n}(K)$. This implies that $[\omega|_n] \subseteq (\pi^{-1} \circ \varphi_{\omega|_n})(K)$. Thus, for each $\omega \in \Lambda^{\mathbb{N}}$

$$\alpha(\omega) \le \limsup_{n\to\infty} \frac{\log \nu([\omega|_n])}{\log r_{\omega|_n}}.$$

We observe that as $\{\omega \to \log p_{\omega_n}\}$ is an independent and identically distributed sequence of random variables and so we may apply the Law of Large Numbers. Thus, for $\nu$ a.e. $\omega \in \Lambda^{\mathbb{N}}$,

$$\lim_{n\to\infty} \frac{1}{n}\nu[\omega|_n] = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} p_{\omega_k} = \int \log(p_{\eta_0}) d\nu(\eta) = \sum_{i\in\Lambda} p_i \log(p_i) = -H(p).$$

And similarly,

$$\lim_{n\to\infty} \frac{r_{\omega|_n}}{n} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \log(r_{\omega|_k}) = -\sum_{i\in\Lambda} p_i \log(r_i) = -\chi(\Phi, p).$$

This allows us to conclude that for $\nu$ a.e. $\omega \in \Lambda^{\mathbb{N}}$,

$$\alpha(\omega) \le \limsup_{n\to\infty} \frac{\log \nu([\omega|_n]}{\log r_{\omega|_n}} = \frac{-H(p)}{-\chi(\Phi, p)} = s.$$

As we stated early this is equivalent to the result,

$$\underline{\dim}(\mu, x) \limsup_{\delta\searrow 0} \frac{\log \mu(B(x, \delta))}{\log \delta} \le \dim_s(\Phi, p) = s.$$

$\square$

**Corollary 4.12.** A trivial corollary following from the definition of $\dim_H \mu$ (Definition 4.7) of this is our desired inequality:

$$\dim_H \mu \leq \dim_s(\Phi, p).$$

We now begin the proof that $\mu$ is exact dimensional. As previously noted, in the case of a finite IFS, this is known. It follows from results proven in [6] that,

$$\dim \mu = \frac{H(p) - H_\nu(\mathcal{P}|\pi^{-1}(\mathcal{B}))}{\chi(\Phi, p)}.$$

Here $H_\nu(\mathcal{P}|\pi^{-1}\mathcal{B})$ is the conditional entropy of $\nu$ with respect to $\mathcal{P}$ the partition of first generation cylinders $\{[i]\}_{i\in\Lambda}$ given the pull back of Borel subsets of $\mathbb{R}$. We recall the definition below.

**Definition 4.13.** Generally for a probability space $(X, \mathcal{M}, m)$ subalgebra $\mathcal{A}$, and function $f \in L^1(X, \mathcal{M}, m)$ there exists a unique $g \in L^1$ often denoted $E_m(\mathcal{M}|\mathcal{A})$ such that for any $E \in \mathcal{M}$

$$\int_E f dm = \int_E g dm.$$

This is the conditional expectation of $f$ given $\mathcal{A}$. In the case that $f = \chi_E$ (when we find the conditional probability of $E$ given $\mathcal{A}$) $g$ is denoted $m(E|\mathcal{A})$.

**Definition 4.14.** For a countable partition $\mathcal{F} \subset \mathcal{M}$, we write $\mathcal{F}(x)$ for the unique $F \in \mathcal{F}$ such $x \in F$. Then, the conditional entropy of $\mathcal{F}$ given $\mathcal{A}$ (a subalgebra of $\mathcal{M}$) is denoted

$$H_m(\mathcal{F}|\mathcal{A}) = \int -\log m(\mathcal{F}(x)|\mathcal{A}) dm(x).$$

Observe that,

$$H_\nu(\mathcal{P}|\Lambda^\mathbb{N}) = \int -\log \nu(P(\omega)|\Lambda^\mathbb{N}) dm(\omega) = \sum_{i\in\Lambda} -\log(p_i)p_i = H(p).$$

We now introduce a few definitions from Ergodic Theory.

**Definition 4.15.** A quadruplet $(X, \mathcal{F}, \theta, T)$ (shortened if the elements are clear) is said to be a measure preserving system if $(X, \mathcal{F}, \theta)$ is a probability space, $T : X \to X$ is measurable, and $\theta(T^{-1}(F)) = \theta(F)$ for all $F \in \mathcal{F}$.

**Definition 4.16.** A measure preserving system $(X, \mathcal{F}, \theta, T)$ is said to be ergodic if, for every $F \in \mathcal{F}$ such that $T^{-1}(F) = F$ then, $\theta(F) = 0$ or 1.

**Theorem 4.17.** The system $(\Lambda^\mathbb{N}, \nu, \theta)$ is measure preserving and ergodic (Recall $\sigma$ is the left shift map.)

*Proof.* It is clear that $(\Lambda^{\mathbb{N}}, \nu)$ is a probability space and $\sigma$ is measurable. Let $n \in \mathbb{N}$ and $w = i_1 \cdots i_n \in \Lambda^n$ be arbitrary. It suffices to show that $\nu(\sigma^{-1}([w])) = \nu([w])$. Here,

$$\begin{aligned}\nu(\sigma^{-1}[w]) &= \nu\left(\{\omega \in \Lambda^{\mathbb{N}} : \sigma(\omega) \in [w]\}\right) = p_{i_1} \cdot \nu\left(\{\omega \in \Lambda^{\mathbb{N}} : i_1\omega \in [w]\}\right) \\ &= p_{i_1} \cdot \prod_{k=2}^{n} p_{i_k} = p_w = \nu([w]).\end{aligned}$$

□

We may now state our main analogous result for IIFS.

**Theorem 4.18.** Suppose $\Phi$ is an IIFS. Furthermore, assume $H(p)$ and $\chi(\Phi, p)$ are finite, and $\Phi$ satisfies the conditions for the existence and uniqueness of $K_\Phi$ as in previous sections. Then, $\mu$ the unique $(\Phi, p)$ stationary measure is exact dimensional with

$$\dim \mu = \frac{H(p) - H_\nu(\mathcal{P}|\pi^{-1}(\mathcal{B}))}{\chi(\Phi, p)}.$$

To prove this we shall need a few lemmas. Denote, $B_\pi(\omega, Rr_{\omega|_n}) = \pi^{-1}(B(\pi\omega, Rr_{\omega|_n}))$. The following Lemma is a well known result in ergodic theory. A proof may be found in [7]

**Lemma 4.19** (Maker's Ergodic Theorem)**.** Let $(X, \mathcal{F}, \theta, T)$ be a measure preserving ergodic system and $\{f_n\}_{n\in\mathbb{N}} \subset L^1$ a pointwise convergent sequence for $\theta$ a.e. $x \in X$ and such that $\sup_{n\geq 1}|f_n| \in L^1$. Then, for $\theta$ a.e. $x \in X$,

$$\lim_{n\to\infty} \sum_{k=0}^{n-1} f_{n-k}(T^k(x)) = \int f d\theta.$$

□

What follows are three technical lemmas regarding the disintegration of $\mathcal{P}$ with respect to $\pi^{-1}(\mathcal{B})$. We denote this set of measures $\{\nu_\omega\}_{\omega\in\Omega}$ where $\Omega \subset \Lambda^{\mathbb{N}}$ with $\nu(\Omega) = 1$. The existence and properties of these $\nu_\omega$ follow from the Lemma 1 of Arnold Faden's [8]. Also, let $R = \operatorname{diam}(K_\Phi)$ and for simplicity of notation we write

$$D(\omega, n) = B_\pi(\omega, Rr_{\omega|_n}).$$

The following Lemma is due to Feng and Hu [6] Proposition 3.5

**Lemma 4.20.** For $\omega \in \Omega$ (as in Lemma 1.19) write

$$g(\omega) = \sup_{\delta>0} \log \frac{\nu B_\pi(\omega, \delta)}{\nu(B_\pi(\omega, \delta) \cap \mathcal{P}(\omega))}.$$

Then $g \in L^1(\nu)$.

$\square$

**Lemma 4.21.** For $\nu$-a.e. $\omega \in \Omega$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \frac{\nu(D(\sigma^k\omega, n-k))}{\nu(D(\sigma^k\omega, n-k)) \cap \mathcal{P}(\sigma^k\omega)} = H_\nu\left(\mathcal{P}|\pi^{-1}(B)\right).$$

**Lemma 4.22.** For every $\omega \in \Omega$ and $n \geq 1$,

$$D(\omega, n) \cap \mathcal{P}(\omega) = \sigma^{-1}(D(\sigma\omega, n-1)) \cap \mathcal{P}(\omega).$$

*Proof of Theorem 4.18.* For simplicity of notation we write

$$\alpha = \frac{H(p) - H_\nu(\mathcal{P}|\pi^{-1}(\mathcal{B}))}{\chi(\Phi, p)}.$$

It suffices to show that for $\mu$-a.e. $x \in \mathbb{R}$ the limit

$$\lim_{\delta\searrow 0} \frac{\log \mu(B(x,\delta))}{\log \delta}$$

exists and is equal to $\alpha$. Because $\mu$ is a stationary measure and $\mu = \pi\nu$ it suffices to show that for $\nu$-a.e. $\omega \in \Lambda^{\mathbb{N}}$, the limit

$$\lim_{n\to\infty} \frac{\log \nu(\pi^{-1}B(\pi\omega, Rr_{\omega|_n}))}{\log Rr_{\omega|_n}} = \lim_{n\to\infty} \frac{\log \nu(D(\omega,n))}{\log r_{\omega|_n}}$$

exists and is equal to $\alpha$. We observe that for $\omega \in \Lambda^{\mathbb{N}}$ and $n \in \mathbb{N}$, and arbitrary $x_0 \in K_\Phi$

$$\nu(D(\sigma^n\omega, x_0)) = \nu(\pi^{-1}B(\pi\sigma^n\omega, R)) = 1.$$

Thus,

$$\begin{aligned}
\nu(D(\omega,n)) &= \frac{\nu(D(\omega,n)))}{\nu(D(\sigma^n\omega, 0))} \\
&= \prod_{k=0}^{n-1} \frac{\nu(D(\sigma^k\omega, n-k)}{\nu(D(\sigma^k\omega, n-k) \cap \mathcal{P}(\sigma^k\omega))} \cdot \frac{\nu(D(\sigma^k\omega, n-k) \cap \mathcal{P}(\sigma^k\omega))}{\nu(D(\sigma^{k+1}\omega, n-k-1)}. \qquad (\dagger)
\end{aligned}$$

Using Lemma 4.22,

$$D(\sigma^k\omega, n-k) \cap \mathcal{P}(\sigma^k\omega) = \sigma^{-1}(D(\sigma^{k+1}\omega, n-k-1)) \cap \mathcal{P}(\sigma^k\omega).$$

Thus we may simplify the expression in (†) above to

$$\nu(D(\omega,n)) = \prod_{k=0}^{n-1} \frac{\nu(D(\sigma^k\omega, n-k)}{\nu(D(\sigma^k\omega, n-k) \cap \mathcal{P}(\sigma^k\omega))} \cdot p_{\omega_k}. \qquad (\star)$$

We observe, by the Law of Large numbers that,

$$\lim_{n\to\infty}\frac{1}{n}\log p_{\omega|_n} = -H(p) \text{ and } \lim_{n\to\infty}\frac{1}{n}\log r_{\omega|_n} = -\chi(\Phi, p).$$

Hence, using ($\star$) and Lemma 4.21, for $\nu$-a.e. $\omega \in \Lambda^{\mathbb{N}}$

$$\begin{aligned}
&\lim_{n\to\infty}\frac{\log \nu(D(\omega,n))}{\log r_{\omega|_n}} \\
&\quad = \lim_{n\to\infty}\left(\frac{1}{n}\left(\prod_{k=0}^{n-1}\frac{\nu(D(\sigma^k\omega, n-k)}{\nu(D(\sigma^k\omega, n-k)\cap\mathcal{P}(\sigma^k\omega))}\cdot p_{\omega_k}\right)\right)\cdot\left(\frac{1}{n}\log r_{\omega|_n}\right)^{-1} \\
&\quad = \frac{H_\nu\left(\mathcal{P}|\pi^{-1}(B)\right) - H(p)}{-\chi(\Phi,p)}.
\end{aligned}$$

Multiplying both numerator and denominator by $-1$ yields the desired result. $\square$

We now prove both Lemmas 4.21 and 4.22.

*Proof of Lemma 4.21.* Let $\{\nu_\omega\}_{\omega\in\Omega}$ be the disintegration of $\nu$ with respect to $\pi^{-1}(\mathcal{B})$ as defined in Lemma 4.19. For every $n \in \mathbb{N}$ we define $f_n : \Lambda^{\mathbb{N}} \to \mathbb{R}$ by:

$$f_n(\omega) = \log\frac{\nu(D(\omega,n))}{\nu(D(\omega,n)\cap\mathcal{P}(\omega))}.$$

We observe that,

$$f_n(\omega) = -\log\frac{\nu(D(\omega,n)\cap\mathcal{P}(\omega))}{\nu(D(\omega,n))} = -\log\frac{1}{\nu(D(\omega,n))}\int_{(D(\omega,n))}\chi_{\mathcal{P}(\omega)}d\nu(\omega).$$

Thus, by the Lebesgue Differentiation Theorem, for $\nu$-a.e. $\omega \in \Lambda^{\mathbb{N}}$

$$\lim_{n\to\infty} f_n(\omega) = \lim_{n\to\infty} -\log\frac{1}{\nu(D(\omega,n))}\int_{(D(\omega,n))}\chi_{\mathcal{P}(\omega)}d\nu(\omega) = -\log\nu_\omega(\mathcal{P}(\omega)) = f(\omega).$$

Using Lemma 4.21 and the Dominated Convergence Theorem $f \in L^1$. Using Maker's Theorem, for $\nu$-a.e. $\omega$,

$$\begin{aligned}
\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\frac{\nu(D(\sigma^k\omega,n-k))}{\nu(D(\sigma^k\omega,n-k))\cap\mathcal{P}(\sigma^k\omega)} &= \lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1} f_{n-k}\left(\sigma^k\omega\right) \\
= \int f d\nu = \int -\log\nu_\omega(\mathcal{P}(\omega))d\nu(\omega) &= H_\nu(\mathcal{P}|\pi^{-1}(\mathcal{B})).
\end{aligned}$$

$\square$

*Proof of Lemma 4.22.* Write $s = Rr_{(\sigma\omega)|_{n-1}}$ so $Rr_{\omega|_n} = r_{\omega_0}s$. Then we have the string of equalities:

$$\begin{aligned}
D(\omega, n) \cap \mathcal{P}(\omega) &= \pi^{-1}(B(\pi\omega, Rr_{\omega|_n})) \cap \mathcal{P}(\omega) \\
&= \pi^{-1}(B(\varphi_{\omega_0}\pi\sigma\omega, r_{\omega_0}s)) \cap \mathcal{P}(\omega) \\
&= \pi^{-1}(\varphi_{\omega_0}(B(\pi(\sigma\omega), s))) \cap \mathcal{P}(\omega) \\
&= \{\eta \in \Lambda^{\mathbb{N}} : \pi\eta \in \varphi_{\omega_0}(B(\pi(\sigma\omega), s)), \eta_0 = \omega_0\} \cap \mathcal{P}(\omega) \\
&= \{\eta \in \Lambda^{\mathbb{N}} : \varphi_{\omega_0}\pi\sigma\eta \in B(\pi(\sigma\omega), s), \eta_0 = \omega_0\} \cap \mathcal{P}(\omega). \\
&= \{\eta \in \Lambda^{\mathbb{N}} : \pi\sigma\eta \in B(\pi(\sigma\omega), s), \eta_0 = \omega_0\} \cap \mathcal{P}(\omega) \\
&= \sigma^{-1}\pi^{-1}B(\pi\sigma\omega, Rr_{(\sigma\omega)|_{n-1}}) \cap \mathcal{P}(\omega). \\
&= \sigma^{-1}(D(\sigma\omega, n-1)) \cap \mathcal{P}(\omega)
\end{aligned}$$

□

Now this result of exact dimensionality does not provide an efficient method for computing the dimension. However, $\mu$ being exact dimensional allows us to use different theory to exploit properties of $\Phi$ and find an effective result for $\dim \mu$. There is a simple case in which $\dim \mu$ can be shown to be the natural candidate, $\frac{H(p)}{\chi(\Phi,p)}$.

**Definition 4.23.** An IFS $\Phi$, either finite or infinite, is said to satisfy the strong separation condition (SSC) if $\varphi_i(K_\Phi)$ are pairwise disjoint for $i, j \in \Lambda$. Thus,

$$K_\Phi = \sqcup_{i \in \Lambda} \varphi_i(K_\Phi).$$

**Corollary 4.24.** If $\Phi$ satisfies the SSC then,

$$\dim \mu = \frac{H(p)}{\chi(\Phi, p)}.$$

*Proof.* The proof of this is quite simple. Let $w \in \Lambda^*$. $\pi([w]) \subset K_\Phi$ is a compact subset, thus it is in $\mathcal{B}$. By the SSC assumption $[w] = \pi^{-1}(\pi([w])$ and so $[w] \in \pi^{-1}(\mathcal{B})$. Thus, $\pi^{-1}(\mathcal{B})$ is the Borel $\sigma$-algebra of $\Lambda^{\mathbb{N}}$ meaning $H_\nu(\mathcal{P}|\pi^{-1}(\mathcal{B})) = 0$. The result follows from Theorem 4.18 □

Even if an IIFS satisfies the SSC, $\inf_{i \neq j \in \Lambda} d(K_i, K_j)$ may be equal to zero. In the case of an IFS, this will be nonzero, and one may prove the previous result without the heavy machinery of 4.18.

**Example 4.25.** Let $t \in (0, 1)$ $p$ a prime number so that $t = 0.t_1t_2t_3\ldots$ as its base-$p$ expansion. For each $n \in \mathbb{N}$, let $s_n = t_1 \ldots t_n$ and let $E_t$ be the set of all real numbers in $(0, 1)$ with base-$p$ expansion $0.s_is_js_k\ldots$ where $\{i, j, k, \ldots\}$ is an arbitrary sequence of natural numbers. We have an IIFS:

$$\Phi = \{\varphi_i(x) = 0.s_i + xp^{-i-1}\}_{i \in \mathbb{N}},$$

and $K_\Phi = E_t$.

We show that $\Phi$ satisfies the SSC and calculate $\dim \mu$. Let $u \in (0,1)$ with base-$p$ expansion $u = 0.u_1u_2u_3\ldots$ such that there exists $q \in \{0,\ldots,p-1\}$ with $u_n \neq q$ for all $n \in \mathbb{N}$. Then, set $t = qp^{-1} + up^{-2}$. If $\varphi_i(x) = \varphi_j(x)$ for some $j > i \in \mathbb{N}$, $x \in (0,1)$ then, the $t_{i+1} = q$, which is a contradiction. Thus, $\Phi$ satisfies the SSC.

Let $(k_n = 2^{-n})_1^\infty$ be a probability vector, $\nu = k^{\mathbb{N}}$ the Bernoulli measure on $\mathbb{N}^{\mathbb{N}}$ and $\mu$ the $(\Phi, k)$ invariant measure. Then,

$$\dim \mu = \frac{H(k)}{\chi(\Phi,k)} = \frac{-\sum_1^\infty 2^{-n}\log 2^{-n}}{-\sum_1^\infty 2^{-n}\log p^{-i-1}} = \frac{3\log(2)}{3\log(p)} = \frac{\log(2)}{\log(p)}.$$

Because $\Phi$ has algebraic parameters and no exact overlaps we may use the result of 2.8 to calculate $\dim K_\Phi$. It suffices to find $\dim_s \Phi$ that is $s \in (0,\infty)$ such that $\sum_2^\infty p^{-is} = 1$. This is equivalent to, $p^{-s} + p^{-2s} = 1$. Thus, $p^{-s}$ is a root to the equation $x^2 + x - 1 = 0$. Setting $\phi = (\sqrt{5}-1)/2$ the golden number then:

$$\dim K_\Phi = -\frac{\log \phi}{\log p}.$$

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